RESECTION COMPUTATIONS IN TRIMBLE ACCESS™

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Introduction

There are two methods of computing a resection in the Trimble® Access™ software - the Standard and Helmert methods. The following sections outline the formulae that are used in these resection computations.

NOTE – *In all the following formulae, x or X values refer to coordinate values on the east-west axis and y or Y values refer to coordinate values on the north-south axis.*

Standard resection

Calculation principle

An initial estimate of the instrument station position is computed. Observation equations are formed for the observations to each control point.

The observations to the control points do not need to have angles and distances observed. If no distances are measured then angular observations to a minimum of three control points are required.

A matrix-based least squares adjustment is carried out using the formed observation equations to determine the instrument station coordinates, orientation correction (Face 1 and Face 2) and scale (if not fixed).

Step 1: Initial Estimate of Instrument Station Position

The routines will compute an estimated position for the instrument station using either a classical three point angular resection or observations to two control points with measured distances, depending on which can be computed first.

The estimated position for the classical three point angular resection is computed using the following formulae:

[image]

with: 3 observed control points 1, 2 and 3 (coordinate values X_1, Y_1, X_2, Y_2, X_3, Y_3)

> the angle α observed between points 1 and 2
>
> the angle β observed between points 2 and 3
>
> the azimuth az_1 and distance L_1 from point 2 to point 1
>
> the azimuth az_2 and distance L_2 from point 2 to point 3

$$\Phi = 360° - (\alpha + \beta + (az_2 - az_1))$$

$$\omega = \tan^{-1}\left(\frac{\sin(\Phi)}{\cos(\Phi) + \frac{L_2 \times \sin(\alpha)}{L_1 \times \sin(\beta)}}\right)$$

$$l = \frac{L_2 \times \sin(180° - (\beta + \omega))}{\sin(\beta)}$$

$$az = az_2 - \omega + 180°$$

$$X_{est} = X_3 + l \times \sin(az)$$

$$Y_{est} = Y_3 + l \times \cos(az)$$

The estimated position for observations to two control points with measured distances is computed using the following formulae:
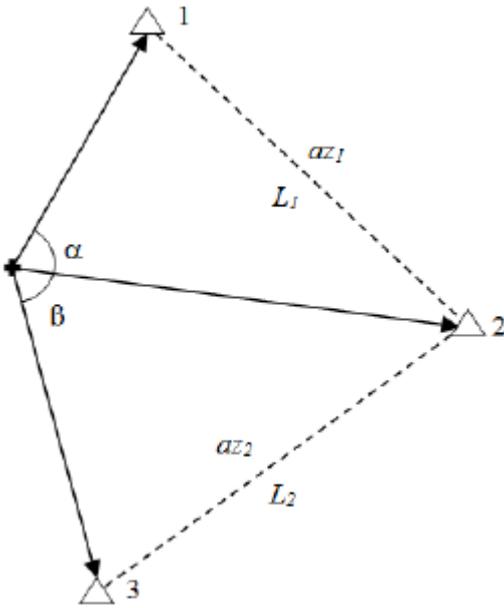
with: 2 observed control points 1 and 2 (coordinate values X_1, Y_1, X_2, Y_2)

> the angle f computed at point 2 (the angle between the baseline from point 2 to point 1 and the direction from point 2 to the occupied point)
>
> the azimuth az_1 and distance L_1 from point 2 to point 1 the azimuth az from point 2 to the occupied point
>
> the horizontal distance L_2 measured to point 2 the horizontal distance L_3 measured to point 1

using the cosine rule:

$$\cos(\Phi) = \frac{L_1^2 + L_2^2 - L_3^2}{2 \times L_1 \times L_2}$$

$$az = az_1 - \Phi$$

$$X_{est} = X_2 + L_2 \times \sin(az)$$

$$Y_{est} = Y_2 + L_2 \times \cos(az)$$

The elevation for the estimated position is computed using the first observation to a control point with an elevation value for which a vertical distance can be computed and then applied to the control point elevation.

Step 2: Build the observation equations

The observation equation that is used to fill the observation matrix for the least squares adjustment is:

$$V = \omega_{F_1} - \omega_{F_2} - Xterm - Yterm - Zterm - Scaleterm - (Obs - Calc)$$

with: Correction value V

Face 1 orientation value ω_{F1}

Face 2 orientation value ω_{F2}

Observed value 'Obs'

Calculated value 'Calc'

Note that all observations on both Face 1 and Face 2 are passed into the observation equations, without initial meaning of the Face 1 and Face 2 observations to the same points. This means that independent orientation corrections are computed and maintained for each face on the instrument. The independent Face 1 and Face 2 orientation corrections effectively encapsulate any residual collimation on the instrument. All following Face 1 observations at the resected station have the Face 1 orientation correction applied to them, and following Face 2 observations have the Face 2 orientation correction applied to them. This mechanism allows Face 2 observations, with no matching Face 1 observations, to be handled without making assumptions about the instrument collimation.

In the following sections X, Y and Z are used to identify the east, north, and elevation values of a given resection control point. X_{est}, Y_{est} and Z_{est} are used to identify the estimated instrument position coordinate values. Observation equations are created for every individual observation, thus allowing the independent computation of Face 1 and Face 2 orientation values if observations on both faces are observed. The scale value is not computed if a fixed scale is specified.

Horizontal directions

For horizontal directions the observation equation values are as follows:

ω_{F1}	=	1 for a Face 1 observation, 0 for a Face 2 observation
ω_{F2}	=	1 for a Face 2 observation, 0 for a Face 1 observation
X term	=	$-\left(\dfrac{X - X_{est}}{HD^2}\right)$
Y term	=	$\left(\dfrac{Y - Y_{est}}{HD^2}\right)$
Z term	=	0 (horizontal direction does not provide any Z component data)

Scale term	=	0 (horizontal direction does not provide any scale component data)
Obs	=	Observed direction to control point
Calc	=	$\tan^{-1}\left(\dfrac{X - X_{est}}{Y - Y_{est}}\right)$ with control point coordinate values X, Y
HD	=	calculated horiz. distance to control point

Slope distances/vertical angles

For slope distance/vertical angle data horizontal distance and vertical distance observation equation are constructed.

Horizontal distances

For horizontal distances the observation equation values are as follows:

ω_{F1}	=	0 (the horizontal distance does not provide any orientation component data)
ω_{F2}	=	0 (the horizontal distance does not provide any orientation component data)
X term	=	$\left(\dfrac{X - X_{est}}{HD}\right) \times SF$
Y term	=	$\left(\dfrac{Y - Y_{est}}{HD}\right) \times SF$
Z term	=	0 (horizontal direction does not provide any Z component data)
Scale term	=	HD
Obs	=	$SD \times \sin(VA)$
Calc	=	$\sqrt{(X - X_{est})^2 + (Y - Y_{est})^2} \times SF$ with control point coordinate values X, Y

Vertical distances

For vertical distances the observation equation values are as follows:

ω_{F1}	=	0 (the vertical distance does not provide any orientation component data)
ω_{F2}	=	0 (the vertical distance does not provide any orientation component data)
X term	=	0 (vertical distance does not provide any X component data)
Y term	=	0 (vertical distance does not provide any Y component data)
Z term	=	1

Scale term	=	0 (vertical distance does not provide any scale component data)

Obs	=	$SD \times \cos(VA) + IH - TH$

IH = Instrument Height

TH = Target Height

SD = Slope distance with corrections for prism constant, atmospherics and projection applied

VA = Vertical angle with curvature and refraction corrections applied

Calc	=	$Z - Z_{est}$ with control point coordinate values X, Y

Vertical angles

For vertical angles the observation equation values are the same as for vertical distances with distance terms being calculated distances:

ω_{F1}	=	0 (the vertical distance does not provide any orientation component data)
ω_{F2}	=	0 (the vertical distance does not provide any orientation component data)
X term	=	0 (vertical distance does not provide any X component data)
Y term	=	0 (vertical distance does not provide any Y component data)
Z term	=	1
Scale term	=	0 (vertical distance does not provide any scale component data)

Obs	=	$SD \times \cos(VA) + IH - TH$

IH = Instrument Height

TH = Target Height

SD = Calculated slope distance to control point with corrections for prism constant, atmospherics and projection applied

VA = Vertical angle with curvature and refraction corrections applied

Calc	=	$Z - Z_{est}$ with control point coordinate values X, Y

Step 3: Determine the observation weighting

Horizontal directions

The weighting of the horizontal directions is computed as follows:

$$w = \frac{1}{HA_{prec}^2 + \left(\frac{InstCent_{err}}{HD}\right)^2 + \left(\frac{BSCent_{err}}{HD}\right)^2}$$

with:

	HA_{prec}	=	Instrument horizontal angle precision
	$InstCent_{err}$	=	Station setup centering error
	$BSCent_{err}$	=	Backsight centering error
	HD	=	Calculated horizontal distance to control point

Horizontal distances

The weighting of the horizontal distances is computed as follows:

$$DistE\rho Sqd \quad = \quad ((EDMprecision + EDMppm \times ObsSID) \times \sin(ObsVA))^2 \quad +$$

$$(ObsSID \times \cos(ObsVA) \times VA_{prec})^2$$

$$w = \frac{1}{DistErrSqd + InstCent_{err}^2 + BSCent_{err}^2}$$

with:

	$ObsSID$	=	Observed slope distance
	$ObsVA$	=	Observed vertical angle
	VA_{prec}	=	Instrument vertical angle precision
	$InstCent_{err}$	=	Station setup centering error
	$BSCent_{err}$	=	Backsight centering error

Vertical distances

The weighting of the vertical distances is computed as follows:

$$StdError = HD \times 0.00005$$

(0.00005 = 50mm per kilometre – an empirical allowance for the lower accuracy of vertical angles due to unknowns such as refraction)

$$VAprecisionTerm = HD \times InstVAprecision \times 4.84814E - 6$$

(4.84814E-6 = 1 second of arc expressed in radians)

NOTE – Any horizontal distances (HD) less than 30m are treated as if they are 30m for weighting purposes (this means that the weighting for all vertical distances, where the horizontal distance is less than 30m, will be the same).

$$w = \frac{1}{StdErr^2 + VAprecisionTerm^2}$$

Vertical angles

The weighting of the vertical angles is computed the same as for vertical distances as follows:

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(0.00005 = 50mm per kilometre – an empirical allowance for the lower accuracy of vertical angles due to unknowns such as refraction)

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NOTE – Any horizontal distances (HD) less than 30m are treated as if they are 30m for weighting purposes (this means that the weighting for all vertical angles, where the horizontal distance is less than 30m, will be the same).

$$w = \frac{1}{StdErr^2 + VAprecisionTerm^2}$$

Where HD is the calculated horizontal distance to the control point.

Step 4: Carry out the matrix computations

The observation equations create the observation matrix *B*. The computed weighting values create the weighting matrix *w*. The *Obs – Calc* values create the matrix *f*.

The corrections to be applied to the estimated instrument position, as well as the orientation and scale values are returned in the corrections matrix Δ.

The following matrix computations are carried out with the *t* suffix indicating a matrix transposition and a -1 suffix indicating a matrix inversion.

$$N = B^t wB$$

$$t = B^t wf$$

$$\Delta = N^{-1}t$$

These computations are carried out iteratively until the corrections matrix values converge (the coordinate corrections have changed by less than 0.0001 and the scale has changed by less than 0.1ppm) or a maximum of 15 iterations has been carried out. Convergence of the orientation values is not monitored for the purposes of terminating the computation. If the results do not converge then the computation is deemed to have failed.

Step 5: Apply the corrections values from the corrections matrix

The corrections matrix Δ contains the following result values:

Easting (X) correction value

Northing (Y) correction value

Elevation (Z) correction value

Face 1 orientation correction

Face 2 orientation correction

Resection setup scale value

The Easting, Northing and Elevation correction values are added to the X_{est}, Y_{est} and Z_{est} values respectively to give the resection station coordinate values. The computed Face 1 and Face 2 orientation values are available directly, as is the scale value if computed.

Step 6: Compute residuals and standard errors

The residuals for all the observations are computed using the observation equation matrix B, the corrections matrix Δ and the *Obs – Calc* matrix f using the following formula:

$$v = f - B\Delta$$

with v being the matrix of residual values.

The sum of the squares of the weighted residuals (SR) is computed using the matrix formula:

$$SR = v^t w v$$

In fact two separate sums of the squares of the weighted residuals are computed by separately summing the horizontal and vertical residuals to give SRHz and SRVt respectively.

The horizontal and vertical redundancies are computed as follows:

For a fixed scale computation:

$$r_{hz} = n_{2D} - 3$$

For a free scale computation:

$$r_{hz} = n_{2D} - 4$$

$$r_{vt} = n_{1D} - 1$$

with: r_{hz}　　　=　　horizontal redundancy

　　　　r_{vt}　　　=　　vertical redundancy

n_{2D}	=	number of observations contributing to the 2D (horizontal) solution
n_{1D}	=	number of observations contributing to the 1D (vertical) solution

If horizontal angular observations have been observed on both faces the horizontal redundancy is reduced by one.

The standard errors of unit weight (s) are computed from the sum of the squares of the weighted residuals and the sum of the weights as follows:

$$\sigma_{Hz} = \sqrt{\frac{SR_{Hz}}{r_{Hz}}} \qquad\qquad \sigma_{vt} = \sqrt{\frac{SR_{vt}}{r_{vt}}}$$

The standard error values for the computed results are calculated using the standard errors of unit weight (s) values above and the appropriate terms from the leading diagonal of the inverted N matrix as described in step 4 above:

$$X_{SE} = \sigma_{hz} \times \sqrt{DiagN^{-1}_{XTerm}}$$

$$Y_{SE} = \sigma_{hz} \times \sqrt{DiagN^{-1}_{YTerm}}$$

$$Z_{SE} = \sigma_{vt} \times \sqrt{DiagN^{-1}_{ZTerm}}$$

$$\omega F1_{SE} = \sigma_{hz} \times \sqrt{DiagN^{-1}_{\omega F1Term}}$$

$$\omega F2_{SE} = \sigma_{hz} \times \sqrt{DiagN^{-1}_{\omega F2Term}}$$

$$SF_{SE} = \sigma_{hz} \times \sqrt{DiagN^{-1}_{SFTerm}}$$

with:	X_{SE}	=	standard error in computed X (East) value
	Y_{SE}	=	standard error in computed Y (North) value
	Z_{SE}	=	standard error in computed Z (Elevation) value
	$\omega F1_{SE}$	=	standard error in computed F1 orientation
	$\omega F2_{SE}$	=	standard error in computed F2 orientation
	SF_{SE}	=	standard error in computed scale factor

$$DiagN_{Term}^{-1}$$ =   the appropriate term from the inverted N matrix leading diagonal

Helmert resection

Calculation principle

Two or more known control points are required – the computation caters for a maximum of 25 known control points. When more than two known control points are used a least squares solution provides the best fit transformation parameters.

Each control point used in the computation must be observed with angles and distances (local coordinates must be able to be computed with respect to the instrument point).

A four parameter Helmert transformation is computed to transform the local coordinates to the known control coordinates, thus providing coordinates for the instrument point in terms of the known control points.

All observations (angles and distances) are given the same weight. The scale factor parameter can be either fixed or free.

As with the Standard resection computation, the Helmert resection also computes both Face 1 and Face 2 orientation corrections when both Face 1 and Face 2 observations are observed. In this case the average collimation is computed from the Face 1/Face 2 observation pairs, and is used to convert the Face 2 observations to equivalent Face 1 observations for use in the Helmert resection computation. This means a Face 1 orientation correction is returned by the resection computation, and the Face 2 orientation correction is then obtained by applying the average collimation to the Face 1 orientation correction. All following Face 1 observations at the resected station have the Face 1 orientation correction applied to them, and following Face 2 observations have the Face 2 orientation correction applied to them.

Step 1: Calculation of local (x, y) coordinates

$$x_i = \sin Hz_i \cdot s_i$$

$$y_i = \cos Hz_i \cdot s_i$$

Step 2: Transformation from local (x, y) into known (X, Y) system

Given:	X_i, Y_i	Known coordinates of control points
	x_i, y_i	Local coordinates of control points
	for 1=1,...,n	Control points – n = number of control points (max 25)
Unknown:	m	Scale factor

$$\omega \qquad \text{Orientation}$$

$$X_0, Y_0$$

Transformation equations:

$$X_i = X_0 + m \cdot \cos \omega \cdot x_i + m \cdot \sin \omega \cdot y_i$$

$$Y_i = Y_0 + m \cdot \cos \omega \cdot y_i - m \cdot \sin \omega \cdot x_i$$

when
$$a = m \cdot \cos \omega \qquad\qquad\qquad \text{Equation 1a}$$

$$o = m \cdot \sin \omega \qquad\qquad\qquad \text{Equation 1b}$$

these transformation formulae become:

$$X_i = X_0 + a \cdot x_i + o \cdot y_i \qquad\qquad\qquad \text{Equation 2a}$$

$$Y_i = Y_0 + a \cdot y_i - o \cdot x_i \qquad\qquad\qquad \text{Equation 2b}$$

with more than two control points the following correction equations apply:

$$v_{X_i} = X_i - (X_0 + a \cdot x_i + o \cdot y_i) \qquad\qquad \text{Equation 3a}$$

$$v_{Y_i} = Y_i - (Y_0 + a \cdot y_i - o \cdot x_i) \qquad\qquad \text{Equation 3b}$$

where:
v_{X_i} = residual in X axis for ith point

v_{Y_i} = residual in Y axis for ith point

Rearranging equations 2a and 2b while summing all the residuals (v_{X_i} and v_{Y_i}) and dividing by the number of control points (n) with [v_{X_i}] and [v_{Y_i}] = 0 gives:

$$0 = -\left(X_0 + a \cdot \frac{[x_i]}{n} + o \cdot \frac{[y_i]}{n} - \frac{[X_i]}{n} \right) \qquad\qquad \text{Equation 4a}$$

$$0 = -\left(Y_0 + a \cdot \frac{[y_i]}{n} - o \cdot \frac{[x_i]}{n} - \frac{[Y_i]}{n} \right) \qquad\qquad \text{Equation 4b}$$

Carrying out the subtractions: Equation 3a - Equation 4a and Equation 3b - Equation 4b gives:

$$v_{X_i} = \left(X_i - \frac{[X_i]}{n} \right) - a \cdot \left(x_i - \frac{[x_i]}{n} \right) - o \cdot \left(y_i - \frac{[y_i]}{n} \right) \qquad \text{Equation 5a}$$

$$v_{Y_i} = \left(Y_i - \frac{[Y_i]}{n} \right) - a \cdot \left(y_i - \frac{[y_i]}{n} \right) - o \cdot \left(x_i - \frac{[x_i]}{n} \right) \qquad \text{Equation 5b}$$

Computation of centroid coordinates (center of gravity):

$$x_S = \frac{[x_i]}{n} \qquad y_S = \frac{[y_i]}{n} \qquad X_S = \frac{[X_i]}{n} \qquad Y_S = \frac{[Y_i]}{n}$$

$$x_i{}' = x_i - x_S \qquad y_i{}' = y_i - y_S \qquad X_i{}' = X_i - X_S \qquad Y_i{}' = Y_i - Y_S$$

Substituting the centroid formulae into equations 5a and 5b results in the new correction equations:

$$v_{X_i} = X_i{}' - a \cdot x_i{}' - o \cdot y_i{}' \hspace{4cm} \text{Equation 6a}$$

$$v_{Y_i} = Y_i{}' - a \cdot y_i{}' + o \cdot x_i{}' \hspace{4cm} \text{Equation 6b}$$

Step 3: Adjustment

Expressing the correction equations using matrix conventions:

$$\underline{v} = \underline{A} \cdot \underline{x} - \underline{l}$$

with:

$\underline{v}^T = \begin{bmatrix} v_{X_1} & v_{Y_1} & v_{X_2} & v_{Y_2} \dots \end{bmatrix}$ \hspace{2cm} Residuals (corrections)

$\underline{l}^T = \begin{bmatrix} -X_1{}' & -Y_1{}' & -X_2{}' & -Y_2{}' \dots \end{bmatrix}$ \hspace{1cm} Absolute part

Matrix of coefficients

$$\underline{A} = \begin{vmatrix} -x_1{}' & -y_1{}' \\ -y_1{}' & +x_1{}' \\ \vdots & \vdots \\ -x_n{}' & -y_n{}' \\ -y_n{}' & -x_n{}' \end{vmatrix}$$

$\underline{P} = \underline{E} = \underline{1}$ \hspace{2cm} Matrix of weights (not needed since all weights equal 1)

$x^T = \begin{bmatrix} a & o \end{bmatrix}$ \hspace{2cm} Unknown transformation of parameters

results in the Normal equation:

$$\underline{A}^T \underline{A} \cdot \underline{x} - \underline{A}^T \underline{l} = 0$$

or

$$\begin{bmatrix} x_i{}^{\prime 2} + y_i{}^{\prime 2} \end{bmatrix} \cdot a \hspace{2cm} = \hspace{2cm} [X_i{}'x_i{}'] + [Y_i{}'y_i{}']$$

$$\begin{bmatrix} x_i{}^{\prime 2} + y_i{}^{\prime 2} \end{bmatrix} \cdot o \hspace{2cm} = \hspace{2cm} [X_i{}'x_i{}'] - [Y_i{}'y_i{}']$$

This resolves to:

$$a = \frac{[X_i' x_i'] + [Y_i' y_i']}{[x_i^2 + y_i^2]} \qquad\qquad o = \frac{[X_i' y_i'] - [Y_i' x_i']}{[x_i^2 + y_i^2]}$$

Step 4: Calculation of the Transformation parameters

The origin (instrument) coordinates result from the use of Equations 2a and 2b:

$$X_0 = X_S - a \cdot x_s - o \cdot y_S$$

$$Y_0 = Y_S - a \cdot y_S + o \cdot x_S$$

The scale factor and orientation values derive from Equations 1a and 1b:

$$m = \sqrt{a^2 + o^2} \qquad\qquad\qquad\qquad \text{Equation 7}$$

$$\omega = \arctan \frac{o}{a}$$

If the scale factor is not to be calculated, then *m* is a known value that must be applied to the parameters *o* and *a* as follows:

$$o' = \frac{o}{m} \qquad\qquad\qquad\qquad a' = \frac{a}{m}$$

Introducing these modified values into Equation 7 results in the computation of the fixed scale factor *m*.

Step 5: Calculation of residuals (corrections)

Use the following equations derived from Equations 6a and 6b:

$$v_{X_i} = X_i - a \cdot x_i' - o \cdot y_i' - X_S \qquad\qquad v_{Y_i} = Y_i - a \cdot y_i' + o \cdot x_i' - Y_S$$

Step 6: Compute the standard deviations

1. Standard deviation of unity weight

$$S_0^2 = \frac{[v_{X_i}^2 + v_{Y_i}^2]}{2n - u}$$

where: n = number of control points

                      u = 3 with a fixed scale factor

                      u = 4 with a free scale factor

2. Standard deviation of instrument point coordinates Y0, X0

$$S_{X_0} = S_{Y_0} = S_0 \sqrt{\frac{1}{n} + \frac{x_{S^2} + y_{S^2}}{[x_i{}^2 + y_i{}^2]}}$$

3. Standard deviation of transformation parameters

$$S_a = S_o = \frac{S_0}{\sqrt{[x_i{}^2 + y_i{}^2]}} = \frac{S_0}{\sqrt{[S_i{}^2]}}$$

4. Standard deviation of the free scale factor

$$m^2 = a^2 + o^2 \quad \Rightarrow \quad 2mdm = 2ada + 2odo$$

$$dm = \frac{a}{m}da + \frac{0}{m}do$$

$$S_m^2 = \left(\frac{a}{m}\right)^2 \cdot S_a^2 + \left(\frac{o}{m}\right)^2 \cdot S_o^2 = \frac{S_0^2}{[S_i^2]} \cdot \frac{a^2 + o^2}{m^2}$$

with $m^2 \approx a^2 + o^2$ results in

$$S_m = \frac{S_0}{\sqrt{[x_i{}^2 + y_i{}^2]}}$$

5. Standard deviation of orientation

$$\tan \omega = \frac{o}{a} \quad \Rightarrow \quad \frac{1}{\cos^2 \omega}d\omega = \frac{a \cdot do - o \cdot da}{a^2} \quad \text{with} \quad \cos^2 \omega = \frac{a^2}{m^2}$$

$$d\omega = \frac{a^2}{m^2}\left(\frac{a \cdot do - o \cdot da}{a^2}\right)$$

$$S_\omega^2 = \frac{1}{m^4} \cdot \left(a^2 \cdot S_o^2 + o^2 \cdot S_a^2\right) = \frac{1}{m^4} \cdot \frac{S_0^2}{[S_i^2]} \cdot \left(a^2 + o^2\right) = \frac{1}{m^2} \cdot \frac{S_0^2}{[S_i^2]}$$

$$S_\omega = \frac{S_o}{m \cdot \sqrt{[S_i^2]}} = \frac{S_o}{m \cdot \sqrt{[x_i{}^2 + y_i{}^2]}}$$